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Identifying decohering paths in closed quantum systems

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Abstract

I discuss a specific proposal for how to identify decohering paths in a "wavefunction of the universe". The emphasis is on determining the correlations among subsystems and then considering how these correlations evolve. The proposal is similar to earlier ideas of Schrödinger and of Zeh, but in other ways it is closer to the "decoherence functional" of Griffiths, Omnès, and Gell-Mann and Hartle. There are interesting differences with each of these which I discuss. In this proposal, once a given coarse-graining is chosen, the candidate paths are fixed in this scheme, and a single well defined number measures the degree of decoherence for each path. The normal probability sum rules are *exactly* obeyed (instantaneously) by these paths regardless of the level of decoherence. I also briefly discuss how one might quantify some other aspects of "classicality". I stress the important role that concrete calculations will play in testing this and other proposals.

1 Introduction

When discussing the whole universe in terms of quantum physics one can not appeal to an outside classical observer. The Copenhagen interpretation of

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quantum mechanics (see for example [1,2,3,4,5]) is meaningless in such situations, and one usually considers some version of the Everett interpretation of quantum mechanics[6,7] . From this point of view the external classical observers of the Copenhagen interpretation become subsystems whose behavior is "sufficiently classical" but whose evolution is none the less described by the evolution of the whole wavefunction. There is no "collapse" of the wavefunction during a measurement, but instead correlations are established between one subsystem and another "apparatus" subsystem.

One way in which quantum mechanics differs from classical mechanics is that it allows quantum interference to occur. Individual "classical paths" or "histories" in general may not be regarded independently of one another. For the Everett picture to work one must be able to identify subsystems which do indeed follow classical paths, and which exhibit a negligible degree of quantum interference. Specific examples which demonstrate how this "decoherence" can occur in quantum systems have been discussed by Joos and Zeh[8], Caldeira and Leggett[9], Zurek[10], and Unruh and Zurek [11], and these authors stress the role of correlations between the decohering systems and the environment in producing decoherence. Still, they do not answer the question: "How can one take a 'wavefunction of the universe' and identify the subsystems which are decohering, and then place a quantitative measure on their degree of decoherence?"

This question is of particular interest in cosmology, where one is lead to view our classical world as having emerged from a completely quantum epoch, in which there may have been initially *no* classical subsystems around (see, for example, the pioneering work of Hawking, Hartle and Vilenkin [12,13,14]). It is likely that a better understanding of the emergence of classical behavior from a fundamentally quantum world will improve our understanding of the origins of the universe, and this perspective has caused increased interest in the phenomenon of decoherence. The importance of this issue in cosmology has been emphasized by Joos[15], Zeh[16], Keifer[17], Halliwell[18], Vilenkin[19], and Unruh and Zurek[11].

A general scheme for identifying decohering histories within a wavefunction has been proposed by Griffiths [20], Omnes[21,22,23], and Gell-Mann and Hartle [24], using a "decoherence functional". Gell-Mann and Hartle also stress the importance for these decohering histories to exhibit other classical qualities which decoherence alone does not insure. We observe in

our world not only decoherence, but the validity of simple classical laws in describing the time dependence of the decohering histories.

In another approach, Zeh[25] has advocated the use of the "Schmidt orthogonal form" for identifying "macroscopic" subsystems. Much earlier, Schrödinger[26] had noted that the Schmidt orthogonal form nicely exhibits the correlations that are present between any two subsystems. Although I was not familiar with reference [25] until this work was completed, the scheme presented in this article has many similarities with the one advocated by Zeh, and might be best viewed as a modest modification of it. The main differences include the the way multiple subsystems are treated, and the role played by the "branching" of classical worlds. Also, the discussion in this paper is in terms of "decohering paths", which makes comparison with the decoherence functional approach more direct (see Section 9).

The thrust of the proposal is to first focus attention on "correlations among subsystems". It is such correlations that make up a great deal of the content of physics, from the description of a laboratory experiment to understanding mechanisms of decoherence. The mathematical results of Schmidt can be used to show how any subdivision of the universe into subsystems results in precisely defined correlations being present. The next step, in relating these formal correlations to physical reality, is to ask if these correlations (or "Schmidt paths ") evolve in a regular manner with time (in general they do not). In this picture a "classical domain" is a collection of subsystems whose correlations evolve in a reliable manner with respect to one another. I show that a natural requirement for decoherence, that paths have well defined probabilities, is equivalent to demanding that the correlations evolve *unitarily*.

The emphasis I give to "decohering coarse grained histories" and to the existence of quasi-classical domains independently of "conscious observers" is the same as that of Gell-Mann and Hartle. None the less, there are technical changes which stem from shifting the focus more directly onto the correlations among subsystems. These changes may clarify some important conceptual issues. I also discuss possible weaknesses of the decoherence functional approach which this approach avoids.

The Schmidt paths approach has potential weaknesses of its own, which I discuss. I explain how the proposal must be tested, and I show how one good counter example can discredit this proposal. (I also argue, contrary to the

authors, that a calculation by Joos and Zeh in [8] is not good counter example due to an inappropriate approximation which is made) My view is that both the Schmidt paths and the decoherence functional approaches need to be tested out on some well understood examples. Because of the important role of correlations in physics, the process of testing the Schmidt paths approach should be particulary instructive, no matter what the outcome.

Like the other authors, I consider a "universe" described by a pure state or density matrix whose evolution is described by a Hermitian Hamiltonian. Thus, for this paper I assume a truly classical background space time. I also assume that the Hilbert space is discrete and finite, which simplifies the notation and, for all we know, is actually the case. Thus, I am assuming that the continuous parameters we use to describe the physical world are either well approximated by closely spaced discrete parameters, or are actually approximations themselves of a world which is fundamentally discrete. I work in the Schrödinger picture, and I discuss in Section 7.2 why I feel it is more suitable than the Heisenberg picture when discussing truly closed quantum systems.

The paper is organized as follows: Sections 2 - 5 set up conventions and describe the the proposal. Section 6 shows how the proposal would apply to some familiar physical situations. In Section 7 the proposal is elaborated further, and ways of quantifying other aspects of classicality are briefly discussed. Section 8 describes crucial tests which must be performed, and points out some possible weak points of the proposal. A direct comparison with the "decoherence functional" (including a brief review of this approach) is provided in Section 9, and I discuss some possible weak points of the decoherence functional as well. Section 10 gives my conclusions.

2 Coarse graining and the system/environment distinction

We discuss the world around us in terms of subsystems which are sufficiently isolated from the rest of the universe to assume their own identity. We then speak of the deviations from pure isolation as "interactions". If we think of a basis $\{|i\rangle\}$ which spans the Hilbert space of the universe, the division into

"subsystem" and "rest of the universe" is accomplished by a re-labeling:

$$|i\rangle = |j(i), k(i)\rangle = |j(i)\rangle_s \otimes |k(i)\rangle_e \ \ (\text{for all } i). \tag{1}$$

In this way we think of the whole Hilbert space as a direct product space of a space corresponding to the subsystem (subscript "s") with a space corresponding to the rest of the universe (subscript "e" for "environment"). This division into subsystems is absolutely central to our whole understanding of the world, and in fact we have no way to attach meaning to the un-subdivided "Hilbert space of the universe" $\{|i\rangle\}$ labeled as such.

In practice, of course, we do not stop at dividing the Hilbert space in two, but discuss a multitude of subsystems. The notion of a "degree of freedom" q essentially just involves the identification of a subspace with a basis labeled by q.

If one limits one's attention to one subspace by considering operators which are only non-trivial in that subspace,

$$\hat{O} \equiv \hat{O}_s \otimes \hat{I}_e, \tag{2}$$

($\hat{I}_e$ is the identity operator in the environment subspace) then one in effect is working with a density matrix, ρ_s, for that subspace.

$$\rho_s \equiv \mathrm{tr}_e \rho_u = \mathrm{tr}_e \left(|\psi_u\rangle\langle\psi_u| \right) \tag{3}$$

where tr_e represents tracing over the environment subspace. I assume that the universe is in a pure state, $|\psi_u\rangle$, but we shall see in Section 7.3 that the discussion is easily generalized for the mixed state case.

Coarse graining is an important notion in physics. It involves working with quantities which are averaged in some way, so as to exclude information which is irrelevant to the physical problem at hand. Coarse graining is nothing other than a particular case of the subdivision of the universe into subsystems. Typically some "collective coordinate" (such as a field averaged over a spatial region) is one subsystem, whereas the relative coordinates are part of the "environment" which is traced over. The coarse graining of a field value into ranges of field values can be thought of as taking a subspace spanned by $\{|\Phi\rangle \mid \Phi \in [-\infty, \infty]\}$ and subdividing it further by taking $|\Phi\rangle \equiv |i(\Phi)\rangle \otimes |\theta(\Phi)\rangle$ where $i(\Phi)$ and $\theta(\Phi)$ are the integer and fractional parts

of Φ in some units. When the θ subspace is traced over one has achieved a coarse graining into ranges of Φ values. There are many examples of divisions into subsystems which are not normally called coarse grainings, but formally the two are the same and I will, for the most part, not make any distinction for this article.

Advocates of the decoherence functional discuss coarse graining in time as well, which does not fall into the general "division into subsystems" framework. I will not use this type of coarse graining in my discussion and I will deal with the qualitative issues addressed by coarse graining in time in another manner. In Section 9 I will discuss this question further, and express concerns that the use of explicit coarse graining in time can cause some serious problems.

3 Correlations between subsystems and the Schmidt orthogonal basis

In order for a subsystem to "be in a pure state", one conventionally writes the wavefunction of the universe in product form

$$|\psi_u\rangle = |f\rangle_s \otimes |g\rangle_e. \tag{4}$$

This form means there are no correlations between the system and environment. It is a very special case for (4) to be true, since one might expect a "typical" wavefunction to have mostly non-zero coefficients in the expansion

$$|\psi_u\rangle = \sum_{j,k} \alpha_{jk} |j\rangle_s |k\rangle_e \tag{5}$$

which would certainly not be equivalent to (4). The fact that we ever get to use wavefunctions of the product form in physics says there is something special about the state of the universe (and the subsystems we have chosen to divide it up into).

There exists a special basis called the Schmidt orthogonal basis[27,26] in which the expansion of a general wavefunction looks simpler than (5). The Schmidt orthogonal basis for a subsystem is none other than the eigenbasis of ρ_s, which being Hermitian, can always be diagonalized producing real

eigenvalues $p^{(j)}$ and the eigenbasis $\{|j\rangle_s^S\}$. (Throughout this paper the superscript "S" indicates a Schmidt orthogonal basis vector.) Likewise, one can construct the density matrix of the environment

$$\rho_e \equiv \mathrm{tr}_s \rho_u = \mathrm{tr}_s (|\psi_u\rangle\langle\psi_u|) \tag{6}$$

and diagonalize it (producing $\{|k\rangle_e^S\}$). The interesting thing is that the eigenvalues of ρ_s and ρ_e are always identical (assuming the universe is in a pure state), with additional zero eigenvalues for the larger matrix. Because of this fact, the direct product states of the two eigenbases form a special basis and the expansion of $|\psi_u\rangle$ in this basis gives

$$\begin{aligned} |\psi_u\rangle &= \sum_i \sqrt{p^{(i)}} \times |i\rangle_s |i\rangle_e \\ &= \sqrt{p^{(1)}} \times |1\rangle_s^S |1\rangle_e^S + \sqrt{p^{(2)}} \times |2\rangle_s^S |2\rangle_e^S + \cdots . \end{aligned} \tag{7}$$

The reason that a general state (Eq (5)) does not look like a product state (Eq (4)) is because in general there are correlations between the subsystem and the environment. The Schmidt orthogonal basis resolves these correlations in a nice way, because each subsystem basis vector is correlated with a unique environment basis vector, as depicted in Eq (7). Note that there are far fewer terms in Eq (7) than in the expansion in a "typical" basis (Eq (5)). The number of terms in the Schmidt expansion equals the size of the smaller subspace, where in the more general case the number of terms is the product of the the two subspace sizes.

In the standard discussions of decoherence [8,9,10,11] much emphasis is placed on the smallness of the off diagonal elements of ρ_s. In particular, the role of correlations between system and environment in suppressing these matrix elements is stressed. For example in [10] Zurek writes something similar to Eq (7) except that the environment states, $|i\rangle_e^S$, are only approximately orthogonal. The sizes of the off diagonal elements of ρ_s are then proportional to the deviations from true orthogonality of the environment states.

It might naively appear that by explicitly diagonalizing ρ_s the off diagonal elements are set to zero with no reference to correlations with the environment. This is definitely not the case, however, since those very correlations help determine ρ_s and thus influence which subsystem basis is the eigenbasis

of ρ_s. In fact, because of the special form of the Schmidt orthogonal expansion (Eq (7)), the Schmidt basis is probably the most concrete way of discussing correlations between a subsystem and its environment.

4 Schmidt paths and the measure of coherence

Although the existence of decoherence is commonly associated with the vanishing of off diagonal elements of ρ_s, that is clearly not the whole story. If it were, one would always have decoherence since one can always diagonalize ρ_s. The problem is that in general the Schmidt basis is redefined in some complicated way at every moment in time as $|\psi_u\rangle$ (and thus ρ_s) evolves. The Schmidt basis will not in general coincide with eigenstates of a practical measurement apparatus. This fact allows the formalism to describe the usual array of quantum interference phenomena, despite one's ability to always diagonalize ρ_s.

As advocates of the decoherence functional have emphasized, it is useful to consider sets of decohering *paths*, which do not interfere among each other. The lack of interference allows one to assign relative probabilities to the different paths which add in the usual way.

As a starting point toward this end I construct what I call "Schmidt paths", which in general are not decohering. I assume that the universe is well behaved enough that the eigenvalues $p^{(i)}$ evolve in a continuous manner with time. This being the case, I can choose the index "i" to remain fixed as $p^{(i)}$ evolves, so $p^{(i)}(t)$ is a continuous function of t. Any realistic degeneracies can be resolved by requiring that the first (and perhaps higher) derivatives are continuous as well. Then the corresponding eigenstates

$$|i, t>_s^S \tag{8}$$

define a path parameterized by t. (I shall mention later the possibility of defining paths in terms of continuity of the Schmidt basis states, instead of the $p^{(i)}$'s, but the idea is much the same)

One expects to be able to assign definite probabilities to decohering paths, and there is certainly nothing about the Schmidt paths which invites this. In

general, the $p^{(i)}$'s might be expected to vary wildly in time, and no particular probability for the whole path would suggest itself. However, if

$$p^{(i)} \approx \text{const.} \tag{9}$$

then $p^{(i)}$ itself it the obvious candidate for the probability for the whole path. More precisely, I define

$$\mathcal{C}^{(i)}(t_1, t_2) \equiv \frac{\Delta p^{(i)}(t_1, t_2)}{\bar{p}^{(i)}(t_1, t_2)}, \tag{10}$$

where $\Delta p^{(i)}(t_1, t_2)$ is defined to be the difference between the maximum and minimum values taken by $p^{(i)}$ in the time interval $[t_1, t_2]$, and $\bar{p}^{(i)}(t_1, t_2)$ is the average value. I intentionally avoid defining $\mathcal{C}$ in terms of $\dot{p}^{(i)}$, since I expect rapid time variation is ok as long as the *amplitude* of the variation is small.

I propose $\mathcal{C}^{(i)}(t_1, t_2)$ as a measure of the coherence between the path labeled by i and the other Schmidt paths over the time interval $[t_1, t_2]$. The smaller $\mathcal{C}$ is, the more completely the path decoheres. This formalism easily accommodates the possibility that there is some Schmidt path defined at all times, which is only decohering over some more limited time range(s).

Requiring that $\mathcal{C}$ be small does not prevent the Schmidt basis from evolving in time. However, to the extent that $\mathcal{C}$ is small, the evolution of the Schmidt orthogonal states is *unitary*, since it is unitary eigenbasis evolution which which corresponds the eigenvalues going unchanged. The nearly unitary evolution of a path with small $\mathcal{C}$ can be regarded as representing the dynamics of that path, although in general there is no reason to expect these dynamics to be simple. The generator of the unitary time evolution will in general be time dependent.

5 Multiple Subsystems

This formalism is easily generalized to the case of multiple subsystems. Starting with the division into two subsystems, as depicted in Eq (7):

$$|\psi_u\rangle = \sqrt{p^{(1)}} \times |1\rangle_1^S |1\rangle_2^S + \sqrt{p^{(2)}} \times |2\rangle_1^S |2\rangle_2^S + \cdots. \tag{11}$$

(with the two subsystems labeled 1 and 2 instead of s and e), one can take any subspace state that appears (say $|1\rangle_2^S$), regard it just as we did $|\psi_u\rangle$, and

start the procedure again:

$$|1\rangle_2^S = \sum_i \sqrt{p_{(1;2)}^{(i)}} \times |i\rangle_{(1;2),1}^S |i\rangle_{(1;2),2}^S \tag{12}$$

or

$$\begin{aligned} |1\rangle_2^S &= \sqrt{\tilde{p}^{(1)}} \times |1\rangle_{\tilde{1}}^S |1\rangle_{\tilde{2}}^S \\ &\quad + \sqrt{\tilde{p}^{(2)}} \times |2\rangle_{\tilde{1}}^S |2\rangle_{\tilde{2}}^S \\ &\quad + \cdots . \end{aligned} \tag{13}$$

The label $(1;2)$ has been added in Eq (12) to identify that this is an expansion of state number 1, subsystem number 2, from Eq (11). Equation (13) is a streamlined expression designed to clarify the basic form of the expansion. The reader should be warned that whenever the tilde ("~") appears (as it does in Eq (13)) the notation has been streamlined for the sake of conceptual clarity. Directly above such an expression will be a technically precise (but perhaps more confusing) form of the same equation.

One can insert Eq (12) into Eq (11) to get

$$|\psi_u\rangle = \sum_i \sqrt{p^{(i)}} \times |i\rangle_1^S \otimes \left[\sum_j \sqrt{p_{(i;2)}^{(j)}} \times |j\rangle_{(i;2),1}^S |j\rangle_{(i;2),2}^S \right] \tag{14}$$

or

$$\begin{aligned} |\psi_u\rangle &= \sqrt{p^{(1)}} \times |1\rangle_1^S \otimes \left[\sqrt{\tilde{p}^{(1)}} \times |1\rangle_{\tilde{1}}^S |1\rangle_{\tilde{2}}^S \right. \\ &\qquad \left. + \sqrt{\tilde{p}^{(2)}} \times |2\rangle_{\tilde{1}}^S |2\rangle_{\tilde{2}}^S + \cdots . \right] \\ &\quad + \sqrt{p^{(2)}} \times |2\rangle_1^S \otimes \left[\sqrt{\tilde{p}^{(1)}} \times |1\rangle_{\tilde{1}}^S |1\rangle_{\tilde{2}}^S \right. \\ &\qquad \left. + \sqrt{\tilde{p}^{(2)}} \times |2\rangle_{\tilde{1}}^S |2\rangle_{\tilde{2}}^S + \cdots . \right] \\ &\quad + \cdots . \end{aligned} \tag{15}$$

or, multiplying everything out:

$$|\psi_u\rangle = \sum_{i,j} \left(\sqrt{p^{(i)}} \sqrt{p_{(i;2)}^{(j)}} \right) \times |i\rangle_1^S |j\rangle_{(i;2),1}^S |j\rangle_{(i;2),2}^S \tag{16}$$

or

$$
\begin{aligned}
|\psi_u\rangle \;=\; & \left(\sqrt{p^{(1)}}\sqrt{\tilde{p}^{(1)}}\right) \times |1\rangle_1^S |1\rangle_{\bar{1}}^S |1\rangle_{\bar{2}}^S \\
& + \left(\sqrt{p^{(1)}}\sqrt{\tilde{p}^{(2)}}\right) \times |1\rangle_1^S |2\rangle_{\bar{1}}^S |2\rangle_{\bar{2}}^S \\
& + \cdots \\
& + \left(\sqrt{p^{(2)}}\sqrt{\tilde{p}^{(1)}}\right) \times |2\rangle_1^S |1\rangle_{\bar{1}}^S |1\rangle_{\bar{2}}^S \\
& + \left(\sqrt{p^{(2)}}\sqrt{\tilde{p}^{(2)}}\right) \times |2\rangle_1^S |2\rangle_{\bar{1}}^S |2\rangle_{\bar{2}}^S \\
& + \cdots
\end{aligned}
\tag{17}
$$

The procedure can be repeated many times producing a whole hierarchy of subdivisions. The state labeled $|m\rangle^S_{(i;j),(k;l),n}$ comes from taking the i^{th} Schmidt orthogonal state for subsystem j in the first subdivision, subdividing further and taking the k^{th} Schmidt orthogonal state of subsystem l, and after one final subdivision, taking the m^{th} state of the n^{th} subsystem. Correspondingly there will be the eigenvalue labeled $p^{(m)}_{(i;j),(k;l)}$.

Each term in Eq (17) may be regarded as a separate Schmidt path. These paths represent histories of more than one subsystem. The probability for each path will just be the product of both of the eigenvalues multiplying that term. In this way each Schmidt path always has an instantaneous probability associated with it.

For these paths to be fully decohering all the probabilities must be constant and both $C^{(i)}$ and $C^{(k)}_{(i;j)}$ must be small. It is possible that the first level of subdivision yields decohering paths while the second level does not. This situation would occur if the p_i's were constant but the $p^{(l)}_{(i;k)}$'s were not. In such a case Eq (17) would be a bad way of thinking about $|\psi_u\rangle$, and this fact would be indicated by having large $C^{(l)}_{(j;k)}$'s. However one could still "back off" one level, and view the wavefunction according to Eq (11). Since the $C^{(i)}$'s are small one still has a perfectly good set of decohering paths. One just has fewer paths, since they are divided up into fewer subsystems.

For each subsystem the trace of the corresponding density matrix is always unity, so the instantaneous probabilities for the Schmidt paths *always* obey the normal sum rules exactly. That means that if one does a "coarse graining" by dropping one low level subdivision, the probability for each

path will be the sum of the probabilities of all the "fine grained" paths which become equivalent when the coarse graining is performed.

It should be remembered that at each level I am making a particular choice of how to make further subdivisions. In general this choice is completely flexible, and any change in the subdivision scheme simply generates a new set of Schmidt paths . Thus the states and eigenvalues really should have an additional label which indicates the particular subdivision scheme used.

6 Some familiar examples

In this section I apply the previous discussion to some familiar physical examples. In doing so, I hope to clarify how the proposal might be expected to work, and what behavior it demands of the Schmidt paths . In Section 8 I turn the question around, and discuss how the behavior described in this section must be explicitly exhibited in calculable systems before one can take the Schmidt paths proposal seriously.

We describe physical "reality" in terms of a classical world. Even our description of quantum mechanics revolves around the behavior of various classical apparatuses in the laboratory. Interpreting a "wavefunction of the universe" amounts to identifying one or more "quasi-classical domains", such as have been discussed by Gell-Mann and Hartle. These domains are sets of subsystems which behave "sufficiently classically". The first requirement for a subsystem to be sufficiently classical is that its path be sufficiently decohering. That property is the subject of this paper. In addition, these paths must have a high degree of regularity and predictability if they are to describe our classical world.

In this Section I will assume the Schmidt paths scheme presented above for the identification of decohering paths is correct. I will also assume that we have available for inspection a wavefunction, $|\psi)$, for a large closed system in which familiar physical situations arise. That is, there is a quasi-classical domain for this system that takes on a familiar form. For example, there might be two subsystems that we would recognize as two billiard balls by the way they interact with one another, and perhaps even a third subsystem which we would call a billiards table, due to the nature of its interactions

with the first two subsystems. We would also want something like a double slit experiment which would allow us to examine the interface between the quantum and classical worlds.

6.1 Two billiard balls

To start with, I consider the simplest case of two billiard balls. First, as in Eq (7), we subdivide the Hilbert space into the first billiard ball subsystem (subscript "$b1$") and the "rest of the universe", or environment:

$$|\psi\rangle = \sqrt{p^{(1)}} \times |1\rangle^S_{b1}|1\rangle^S_e + \sqrt{p^2} \times |2\rangle^S_{b1}|2\rangle^S_e + \cdots . \tag{18}$$

The number of terms in this sum in general could equal the size of the smaller of the two subspaces, which is probably going to be the billiard ball subspace. I really only want the $b1$ subspace to represent the center of mass coordinate, so if $|\psi\rangle$ were really "true to life", there would be subspaces corresponding the relative coordinates of the individual atoms of the billiard ball that would be included in the "environment" subspace. To satisfy the decoherence requirement the $C^{(i)}$'s must be very small over the time period of interest.

One would like to identify a particular basis of the $b1$ subspace as "position". Let us call it $\{|\vec{x_k}\rangle_{b1}\}$, where I am being true to my "discrete and finite" conventions by giving position a discrete index "k". One can then define the operator

$$\hat{x} \equiv \sum_k |x_k\rangle x_k \langle x_k| \tag{19}$$

and the position

$$x^i_{b1}(t) \equiv {}^S_{b1}\langle t, i|\hat{x}|i, t\rangle^S_{b1} \tag{20}$$

(Note that i labels Schmidt states and k labels positions.) As I have discussed, the time evolution of $|i,t\rangle^S_{b1}$ gives the dynamics which determine $x^i_{b1}(t)$. In fact, when our system successfully describes a classical billiard ball, the $|i,t\rangle^S_{b1}$'s should be highly localized in the position basis, and for practical purposes we may regard the $x^i_{b1}(t)$'s as representing the Schmidt paths .

For the system $\imath$ to describe a classical billiard ball, $x^i_{b1}(t)$ should obey Newton's laws. In particular

$$\vec{F} = m\vec{a} \tag{21}$$

should hold, where $\vec{a}$ is given by $\ddot{x}^i_{b1}(t)$. Of course, this will always be the case if we define $\vec{F}$ suitably, and we could always attribute this force to the action of the environment on the billiard ball. However, one of the important properties of our classical world is that forces can be attributed to other classical subsystems interacting in some regular way.

At this point we further subdivide each Schmidt state of the environment subspace into billiard ball number two ($b2$) and the rest of the original environment subspace (e')

$$|i\rangle^S_e = \sum_j \sqrt{p^{(j)}_{(i;e)}} \times |j\rangle^S_{(i;e),b2} |j\rangle^S_{(i;e),e'} \tag{22}$$

or

$$\begin{aligned} |i\rangle^S_e &= \sqrt{\tilde{p}^{(1)}} \times |1\rangle^S_{\tilde{b}2} |1\rangle^S_{\tilde{e}'} \\ &\quad + \sqrt{\tilde{p}^{(2)}} \times |2\rangle^S_{\tilde{b}2} |2\rangle^S_{\tilde{e}'} + \cdots . \end{aligned} \tag{23}$$

just as was done in Eq (13). The whole wavefunction may now be written

$$|\psi\rangle = \sum_{i,j} \left(\sqrt{p^{(i)}} \sqrt{p^{(1)}_{(1;e)}} \right) \times |i\rangle^S_{b1} |j\rangle^S_{(i;e),b2} |j\rangle^S_{(i;e),e'} \tag{24}$$

or

$$\begin{aligned} |\psi\rangle &= \left(\sqrt{p^{(1)}} \sqrt{\tilde{p}^{(1)}} \right) \times |1\rangle^S_{b1} |1\rangle^S_{\tilde{b}2} |1\rangle^S_{\tilde{e}'} \\ &\quad + \left(\sqrt{p^{(1)}} \sqrt{\tilde{p}^{(2)}} \right) \times |1\rangle^S_{b1} |2\rangle^S_{\tilde{b}2} |2\rangle^S_{\tilde{e}'} \\ &\quad + \cdots \\ &\quad + \left(\sqrt{p^{(2)}} \sqrt{\tilde{p}^{(1)}} \right) \times |2\rangle^S_{b1} |1\rangle^S_{\tilde{b}2} |1\rangle^S_{\tilde{e}'} \\ &\quad + \left(\sqrt{p^{(2)}} \sqrt{\tilde{p}^{(2)}} \right) \times |2\rangle^S_{b1} |2\rangle^S_{\tilde{b}2} |2\rangle^S_{\tilde{e}'} \\ &\quad + \cdots \end{aligned} \tag{25}$$

where each term represents a particular path involving both billiard balls. As in the previous section, the equations with tildes are streamlined versions of the precise equations which appear above them.

The $b2$ subspace should also have a position basis, producing $x^j_{(i;e),b2}(t)$ and Newton's laws should be obeyed here as well. For now, let us assume

that each billiard ball only experiences forces attributable to collisions with the other ball. Let the interaction take the simple form of the elastic hard sphere interaction at some fixed ball radius. One crucial requirement is that for each path of $b1$ there is well defined position for $b2$, so that the occurrence of a collision is completely well defined. This feature is provided by the fact that each Schmidt path, or term in Eq (25), is composed of paths for both of the two billiard ball subsystems.

For example, in the first term of Eq (25) $b1$ is on path 1, which is correlated with $b2$ on *its* path 1. Then one can ask: "Do the two paths, $x^1_{b1}(t)$, and $x^1_{(1,e),b2}(t)$, represent the motion of billiard balls obeying Newton's laws and only experiencing forces due to their mutual interaction?". If the answer is "yes" , then one has identified two classical billiard balls within $|\psi\rangle$.

Classical measurement

We see here the essential ingredients of a classical measurement. The velocity $\dot{x}^1_{b1}$ will be constant except at the moments when a collision with $b2$ occurs. One could look at the instants when the velocity changes and "measure" the presence of the second billiard ball, centered two ball radii away. Further study of the vector change in $\dot{x}^1_{b1}$ would reveal more about the second ball. Note that this classical "measurement" involves exploiting the predictable dynamics of classical paths. The only way the issue of decoherence appears is to insure that the classical systems remain classical. We will see that decoherence plays a more central role in the case of quantum measurement.

Adding a subsystem

Having come this far, one can easily see how to add a billiard table. Again, the environment (already reduced to e') is now subdivided further into the billiard table subsystem bt and the rest of the environment e''.

$$|\psi\rangle = \sum_{i,j,k} \left(\sqrt{p^{(i)}} \sqrt{p^{(j)}_{(i;e)}} \sqrt{p^{(k)}_{(i;e),(j;e')}} \right) \times |i\rangle^S_{b1} |j\rangle^S_{(i;e),b2} |k\rangle^S_{(i;e),(j;e'),bt} |k\rangle^S_{(i;e),(j;e'),e''} \tag{26}$$

or

$$|\psi\rangle = \left(\sqrt{p^{(1)}} \sqrt{\tilde{p}^{(1)}} \sqrt{\tilde{p}^{(1)}} \right) \times |1\rangle^S_{b1} |1\rangle^S_{b2} |1\rangle^S_{bt} |1\rangle^S_{e''}$$

$$+\left(\sqrt{p^{(1)}}\sqrt{\tilde{p}^{(1)}}\sqrt{\tilde{p}^{(2)}}\right)\times|1\rangle^{S}_{b1}|1\rangle^{S}_{\tilde{b}2}|2\rangle^{S}_{bt}|2\rangle^{S}_{\tilde{e}''}$$
$$+\cdots. \tag{27}$$

The Schmidt orthogonal decomposition insures that on each multiple-subsystem path the three subsystems, $b1$, $b2$, and bt are each correlated with well defined paths for the other two. One can then ask if the paths of $b1$, $b2$, and bt evolve in time so as to describe the simple motions of two billiard balls on a billiard table. If the answer again is "yes", then one has identified this slightly more complex classical world within $|\psi\rangle$.

Discussing the "branches"

This system exhibits many "branches" as represented by terms in Eq (27). It might be the case that only one term or branch represents nice classical behavior for $b1$, $b2$, and bt, or perhaps many do. If the first term in Eq(27) represents familiar classical behavior, one might in particular wonder about the second term, which has the *same* states for the billiard balls, but a different state for the table, orthogonal to the first. It would seem that the path described by the second term is bound to not correspond to any classical world we know. One should remember, however, that there is no need to use the same basis for interpretation on every branch. The second term will probably have a degree of regularity similar to that of the first term, and there could well be another "position basis" for the billiard table which is the "right" one in this case. One would simply need the changes in $x_{b1}(t)$ and $x_{b2}(t)$ (attributed to collisions with the table wall on the first path) to be similarly associated with the proximity of the wall in the new basis for the second path. Specific calculations are necessary to see what happens in these situations for realistic systems.

This is quite different from the point of view expressed by Zeh[25]. He prefers to require that, for example, the second (and other) terms not be present in Eq. (27), so that the multiple classical subsystems are uniquely correlated with one another. I prefer to let all the terms be there, and investigate the consequences. It is possible that the constraints Zeh imposes will turn out to be necessary to allow decoherence to occur. On the other hand, it may turn out that the presence of other terms helps decoherence, even if perhaps all the terms do not represent nice classical paths.

One might ask why have all the "extra baggage" of many terms when all one wants is one classical system. One certainly could require all but one set of the p_i's in Eq (26) to be zero, leaving just the single classical "world". From the classical point of view, the single term version would seem more "economical", since the "extra baggage" was eliminated. From the quantum point of view, one might have a different notion of "economy". Reducing the expansion of $|\psi\rangle$ to a single term requires imposing additional constraints, whereas a nice classical system was already there before any such constraints were imposed. Furthermore, when quantum processes affect the classical world, the "extra" terms are required, as we shall see below.

The multiplicity of terms actually plays an important role in spontaneous symmetry breaking. It allows the full wavefunction to have a symmetry that is not exhibited in the classical domains. For example, the standard wavefunction for matter believed to emerge from an inflationary cosmology is virtually homogeneous in space. The *a priori* probability of finding a galaxy in any location is independent of the location. It is crucial that the inhomogeneities represented in quantum fluctuations "become classical" [28,29,30,31,32]. We see that the translational invariance is broken in our classical world only because the *local* interactions which cause decoherence cause the decohering paths to be localized ones, with the translational symmetry broken. The whole wavefunction can keep its translational symmetry because it is a sum over many terms representing decohering paths localized in different places.

To further illustrate this point, note that we did not write

$$x_{bi}(t) = \langle \psi, t | \hat{x} | \psi, t \rangle \tag{28}$$

to define the path of the billiard ball. If we had, and $|\psi\rangle$ had an inflationary origin, we would have found some triviality like $x(t) = 0$ because of the translational symmetry. Similarly, the multiple branches can be important in more traditional examples of spontaneous symmetry breaking.

In this subsection I have sketched how the presence of a simple quasiclassical domain would be reflected in the Schmidt paths . It basically requires the subsystems in question to be decohering (small $\mathcal{C}$). In addition, substantial demands are made on other aspects of the dynamics to produce simple classical behavior. This example illustrates the essential ingredients of

arbitrarily large quasi-classical domains, which require more numerous well behaved decohering subsystems.

6.2 The double slit experiment

In the previous subsection I gave an illustration of classical behavior from the point of view of the Schmidt paths . Now I would like to give another illustration for a case when the quantum world noticeably affects the classical domain.

I consider the familiar "double slit experiment". One starts with a beam of electrons which is directed toward a barrier. The barrier is impenetrable except for two slits which are present on the scale of the deBroglie wavelength of the electrons. The electrons evolve into a characteristically quantum mechanical state when they pass through the slits, and the state is characterized by quantum interference between electrons diffracting through the different slits. If a detector is placed behind the barrier, the electron counts at different positions exhibit the peculiarities of the electron's quantum state. In particular there are positions where very few electrons are detected, which is where the diffracting electrons "destructively interfere".

Conditions can be achieved where the beam is emitting individual electrons whose wave packets are well separated in space and time. Individual electrons can be detected (or lost) before the next one is emitted. Still, the same interference effects are observed.

I will now sketch how this experiment would look from the point of view of the Schmidt paths . The setup includes a fairly large classical world from which one assembles the various components of the experiment. Thus the formalism must include many decohering subsystems in order to represent this classical world. I will lump all but the most interesting of these subsystems into the "environment" subsystem.

In addition to the environment (e), I identify three other subsystems: The electron (el), the barrier (b), and the detector (d). It will be useful to have the joint electron-detector subsystem, $el\&d$, present at an intermediate level. The joint subsystem is then further subdivided into el and d. (In general, any order of subdivision is possible, and it is up to you choose one that most effectively exposes the physics which is going on) At the "initial" time (t_1)

the total wavefunction can be written

$$|\psi,t_1\rangle = \sum_{i,j,k}\left(\sqrt{p^{(i)}}\sqrt{p^{(j)}_{(i;e)}}\sqrt{p^{(k)}_{(i;e),(j;el\&d)}}\right) \times |i,t_1\rangle^S_b |j,t_1\rangle^S_{(i;e),e'} |k,t_1\rangle^S_{(i;e),(j;el\&d),el} |k,t_1\rangle^S_{(i;e),(j;el\&d),d} \quad (29)$$

or

$$\begin{aligned}|\psi,t_1\rangle = & \left(\sqrt{p^{(1)}}\sqrt{\tilde{p}^{(1)}}\sqrt{\tilde{p}^{(1)}}\right) \times |1,t_1\rangle^S_b |1,t_1\rangle^S_{e'} |1,t_1\rangle^S_{el} |1,t_1\rangle^S_d \\ & + \left(\sqrt{p^{(1)}}\sqrt{\tilde{p}^{(1)}}\sqrt{\tilde{p}^{(2)}}\right) \times |1,t_1\rangle^S_b |1,t_1\rangle^S_{e'} |2,t_1\rangle^S_{el} |2,t_1\rangle^S_d \\ & + \cdots . \end{aligned} \quad (30)$$

I will begin the discussion when a single electron packet decouples from the electron source (which is part of the environment subsystem). At this stage the electron, the barrier, and the detector should all be decohering on at least one Schmidt path, so at least one set of $\{C^{(i)}, C^{(j)}_{(i;e)}, C^{(k)}_{(i;e),(j,el\&d)}\}$ (corresponding to one term in Eq (30)) should all be small. I will focus on just one term in Eq (30), and call it term 1. In reality if there is one term that describes the experiment, there will be numerous others, differing only by the time when the electron leaves the source, for example.

As the electron approaches the barrier, the three systems will each evolve in their own unitary way. The electron will propagate along, the barrier will just sit there, and the detector will sit in some "ready" state. As the electron passes through the barrier, the subsystems will continue to evolve unitarily, and will remain independent from the point of view of decoherence. The effective Hamiltonian describing the evolution of the electron will represent the interaction with the barrier, and cause the electron state to change (unitarily) into the diffracted form.

The "collapse" of the wavefunction

When the diffracted electron wave "packet" reaches the detector things start to change. The interactions between the *el* and *d* subsystems no longer preserve their decohering status. although the joint electron-detector subsystem (*el&d*) can remain decohering from everything else. One would then have to

write

$$|\psi, t_2\rangle = \sum_i \left(\sqrt{p^{(i)}}\sqrt{p^1_{(i;e)}}\right) \times |i, t_2\rangle^S_b |j, t_2\rangle^S_{(i;e),e'} |j, t_2\rangle^S_{(i;e),el\&d} \qquad (31)$$

or

$$\begin{aligned} |\psi, t_2\rangle \;=\; & \left(\sqrt{p^{(1)}}\sqrt{\tilde{p}^{(1)}}\right) \times |1, t_2\rangle^S_b |1, t_2\rangle^S_{\tilde{e}'} |1, t_2\rangle^S_{el\tilde{\&}d} \\ & + \left(\sqrt{p^{(1)}}\sqrt{\tilde{p}^{(1)}}\right) \times |1, t_2\rangle^S_b |2, t_2\rangle^S_{\tilde{e}'} |2, t_2\rangle^S_{el\tilde{\&}d} \\ & + \cdots . \end{aligned} \qquad (32)$$

One could still perform the expansion of the electron and detector subsystem:

$$|j, t_2, \rangle^S_{(i;e),el\&d} = \sum_k \sqrt{p^{(k)}_{(i;e),(j;el\&d)}} \times |k, t_2\rangle^S_{(i;e),(j;el\&d),el} |k, t_2\rangle^S_{(i;e),(j;el\&d),d} \qquad (33)$$

or

$$\begin{aligned} |j, t_2, \rangle^S_{(i;e),el\&d} \;=\; & \sqrt{\tilde{p}^{(1)}} \times |1, t_2\rangle^S_{\tilde{e}l} |1, t_2\rangle^S_{\tilde{d}} \\ & + \sqrt{\tilde{p}^{(2)}} \times |2, t_2\rangle^S_{\tilde{e}l} |2, t_2\rangle^S_{\tilde{d}} \\ & + \cdots . \end{aligned} \qquad (34)$$

At t_1 this expansion would have resulted in further decohering subsystems. But after the electron and the detector start to interact, one expects the $C^{(k)}_{(i;e),(j;el\&d)}$'s to become large, indicating coherence at this level of subdivision. One is then forced to back off one level and settle for the paths represented by Eq (31).

After a period of time, the electron and the detector may again decohere from each other, at least on some paths. Some of the decohering paths will represent the electron missing the detector and just flying by, leaving the detector still in its "ready" (but nothing detected) state. Other terms will represent the electron interacting with the detector and causing the detector to signal an event.

In some interacting cases the electron and detector paths may remain coherent (signaled by some C remaining large). This would occur if that electron became bound somewhere in the detector. Even the terms which represent

the electron going undetected will have a different electron Schmidt orthogonal states than if the detector was not there. That is because the correlations set up with the detector change ρ_{el}, and thus the Schmidt orthogonal basis in which it is diagonal.

There would be no path representing an electron passing through the slits and going undetected which decohered over that entire time (unless the detector and the electron states have *no* interaction on all paths). The decohering paths for the electron simply stop decohering during the "measurement" (when correlations are being established with the detector), and re-emerge again later, in a different form. During this intermediate period the evolution within the individual electron and detector subspaces is non-unitary, even though the combined system can evolve unitarily. This special period, when the electron and detector are lost (as separate entities) to the classical domain would correspond roughly to the period when the wavefunction "collapses" in the Copenhagen point of view.

6.3 Further discussion

At the level of decoherence, the interaction between the electron and the barrier is completely equivalent to the interaction between the two billiard balls. In each case the two subsystems maintain their decoherence, and thus their independent identities. In the case of the billiard balls, the paths continue to maintain the desired level of decoherence. The electron, however, goes on to encounter the detector, and its decoherence is destroyed. If the world beyond the barrier were a strange one, in which detectors interacted coherently with double slit diffraction patterns, then the double slit experiment might look more like the billiard ball problem.

Note that real world billiard balls are not immune to having their decoherence terminated. If a billiard ball rolls into a furnace and is ionized, the original billiard ball subsystem becomes hopelessly coherent as the individual ions interact with the environment.

More about branching

It is interesting to think about the emergence and loss of quantum coherence in terms of "branching" of decohering paths. When the detector and the

electron started interacting in the double slit experiment, the total number of decohering paths decreased. This was because one level of subdivision (or "fine graining") was lost, so there were simply fewer labels to distribute among the decohering paths. Pairs of decohering Schmidt orthogonal paths that were once distinguished from one another by their particular electron and detector paths became lumped together, since the joint electron-detector subsystem (which was still decohering), was able to offer fewer distinctions. This effect might be regarded as the "joining together" of decohering paths. In a similar manner, the emergence of decoherence at a new level of subdivision can be thought of as "branching" of decohering paths. In that case new labels are available, so there are multiple paths carrying identical old labels, which are distinguished from one another by their new labels.

At the end of the day, a double slit experiment may not represent a net increase in the number of branches. An observer may think in terms of an increase in the number of branches because *before* the detection he made sure to identify the states of the electron and detector on his particular branch. This identification remained valuable as long as the electron and the detector remained separately decohering systems, but became less useful during the detection, when they stopped decohering (and the total number of paths decreased). After the measurement, when the number of paths increases again, the best the observer can do with the *old* information is, based on the (decoherent) unitary evolution of the joint detector-electron system, calculate the relative probability that he is on various paths. Of course, he can also observe the detector and identify his path more completely.

This may be a good point to remind the reader that the number of paths is not directly related to the size of the Hilbert space, which of course remains unchanged. Instead it is related to the number of subsystems identified. This is because the Schmidt orthogonal expansion of a state has far fewer terms than the expansion into some arbitrary basis, as discussed in Section 3.

I should also remark that it is by no means necessary to make the *same* division into subsystems on each path (that is, in each term in the expansion of the wavefunction). This is in fact quite important, since, for example, the billiard ball may roll into a furnace on one branch and not on another. On each of those branches different subdivisions would be the most useful ones.

7 Other aspects of the proposal

7.1 General discussion – when is $\mathcal{C}$ small enough?

One is well advised in physics to demand a clear connection between one's formalism and real laboratory situations. It is often the case that ambiguities are resolved (or problems are exposed) when the measurement apparatus, for example, is included in the calculation.

It is when one starts to think this way that I find the Schmidt paths proposal particularly attractive. The entire content of experimental physics (and our existence for that matter!) winds up being a matter of correlations among subsystems. One is always addressing questions like: "How is the state of one detector correlated with the state of another detector or with a clock?", or much more complex versions of the same idea.

The Schmidt paths proposal deals explicitly with correlations among subsystems. The definition of the Schmidt paths is one which most clearly exhibits the correlations among the chosen subsystems. For some of these correlations to have meaning to us, they must develop and evolve in a regular, reliable way. Therefore it is interesting that the decoherence condition (small $\mathcal{C}$) is just the condition that these correlations evolve in a *unitary* manner. The smaller $\mathcal{C}$ is, the more the time evolution of the corresponding Schmidt orthogonal states is *unitary* (for both the "system" and the "rest of the universe" subspaces). Although this unitary evolution might in general be quite complicated, the unitary constraint still brings a certain level of regularity to the evolution of the correlations. To come closer to describing our world the evolution of the Schmidt orthogonal basis must take on a *simple* unitary form, at least relative to the other subsystems with respect to which correlations are important. I will return to this issue briefly in Section 7.4.

One might ask: how small is "small enough" for $\mathcal{C}$? The answer to that lies entirely in the physical situation one wishes to discuss. In describing the classical world around us we find certain correlations to be preserved in an enormously reliable and regular way, such as the correlation between the key in our pocket and the lock on our door. In a formalism that purported to describe that situation, one could in principle calculate the degree of deviation from the known, reliable correlations. One could compare that with the observed bounds and see if the $\mathcal{C}$ in the calculation was small enough.

Our correlations with other subsystems, such as free neutrons, are much less reliable, and a good theory would not let a free neutron subsystem decohere for very long.

7.2 Observables and the Schrödinger vs. Heisenberg pictures

It is common in standard quantum mechanics to think of quantum measurements in terms of "observables" represented by Hermitian operators. In a truly closed system a measurement must just amount to a rearrangement of the correlations among subsystems. A good measurement apparatus would be one for which the Schmidt basis for both the apparatus and the measured subsystem reliably wound up being a very particular basis after the measurement. We would then know that a given detector signal is correlated with a particle being localized in a definite region, for example. In this way the interaction with a measurement apparatus is associated with the "collapse of the wavefunction" onto a particular basis in the subsystem to be measured.

A "predicting subsystem" such as a physicist can predict the outcome of an encounter between another subsystem and a sufficiently well behaved apparatus by just looking at the expansion of the Schmidt state the subsystem has (on *his* path) in the standard basis associated with the apparatus. Some of the physicist's manipulations, such as the calculation of "expected value" of a measurement are easily represented (in the usual way) in terms of a Hermitian matrix who's eignevectors are the special basis vectors associated with the apparatus (such as the position basis, in the case of a screen or detector). This is exactly what we did in Eqs (19) and (20) in order to discuss the position of the billiard ball, although we were not discussing *quantum* measurements of the position there.

The Hermitian matrix, however, is just a handy tool (of great practical value!) whereby the effects of an apparatus subsystem may be represented in a smaller Hilbert space which does not include the apparatus explicitly. In reality one can only attach meaning to $x_{b1}(t)$ (the position of billiard ball number 1), to the extent that one can use it to understand correlations between the billiard ball and an "apparatus", which is just another subsystem of the universe. For example, $x_{b1}(t)$ acquires some meaning when, in addition, one considers the postion of the second billiard ball subsystem ($x^1_{(1,e),b2}(t)$).

Then one can use $x_{b1}(t)$ to predict when collisions will occur, and thus when $\dot{x}^1_{(1,e),b2}(t)$ will change.

In this way, when one is considering the wavefunction of the "universe" or of any truly closed system, one often has no use for operators representing observables. Instead one should just look directly at how the correlations among subsystems are established or changed. Without the presence of operators (aside from the Hamiltonian), there simply is no Heisenberg picture, and for this reason I believe that the Schrödinger picture is best suited for many problems in quantum cosmology.

7.3 Is the universe in a pure state?

Formally, the Schmidt paths proposal works just as well if the state of the universe is represented by a general density matrix ρ_u (or "mixed state") rather than a pure state. One simply starts by diagonalizing this density matrix, and then carries out further subdivisions as before. However, the probability for each path will include a factor which is an eigenvalue of ρ_u. This means that at least one eigenvalue of ρ_u must be as constant as the coefficient of the most highly decohering path. In the limit when the eigenvalue corresponding to our path is absolutely constant, our state would be evolving unitarily in time, and it would be impossible to distinguish between a pure state universe and a mixed state universe.

7.4 Measuring "Classicality"

I have attached considerable importance to the need for more regularity than mere decoherence in order to accurately describe the classical world. (This point has already been made very well by Gell-Mann and Hartle [24].) As Gell-Mann and Hartle have remarked, it would be nice to have some quantitative measure of "classicality" which includes this notion of regularity. Such a measure would not be used to verify the classical behavior of our own world, so much as to answer the question: "do the wavefunction and Hamiltonian of our universe describe other classical domains in addition to ours?" Such classical domains could in principle be radically different from ours, and could consist of a completely different subdivision of the universe into subsystems. By considering this question one might be able to understand which (if any)

aspects of our physical world are dictated by the need for a classical domain. In this section I describe briefly an idea about how to quantify additional aspects of classical behavior.

The unitary evolution of the Schmidt orthogonal basis provided by the decoherence condition is not so constraining because the generator of this evolution can in general have some arbitrary time dependence. The generator of the unitary evolution of the Schmidt orthogonal basis is what one might want to call the effective hamiltonian for that subsystem. One measure of regularity, then, might be the extent to which the effective Hamiltonian depends on time.

More specifically, H^{eff} for a particular subsystem is defined in terms of the evolution of the Schmidt orthogonal basis as follows:

$$ {}^{S}_{s}\langle j,t|H^{eff}_{s}(t)|k,t\rangle^{S}_{s} \equiv \lim_{\Delta t\to 0} \frac{{}^{S}_{s}\langle j,t|k,t+\Delta t\rangle^{S}_{s} - \delta_{jk}}{-i\Delta t} \qquad (35) $$

One simple way to measure the time dependence of H^{eff} is to look at the time dependence of its eigenvalues, $\lambda_i(t)$. In complete analogy with the definition C (Eq (10)) one can define

$$ \mathcal{E}_i(t_1,t_2) \equiv \frac{\Delta\lambda_i(t_1,t_2)}{\bar{\lambda}_i(t_1,t_2)}. \qquad (36) $$

The smaller $\mathcal{E}_i(t_1,t_2)$ is, the less λ varies over the time interval $[t_1,t_2]$. Technically the λ's will have residual imaginary parts since "sufficiently decohering" paths need not have *absolutely* unitary evolution of the Schmidt orthogonal basis. It is perhaps simplest to to separate the two issues and just use the real part of λ_i in Eq (36), counting on the small C condition to keep the imaginary parts small. On the other hand, the size of the imaginary part of λ might provide a very useful handle on the degree of decoherence. It might be possible to develop a more comprehensive discussion of classicality by focusing exclusively on the λ's and requiring them to be both real and constant.

The work of Kübler and Zeh [33], Zeh [25], and Joos and Zeh [8] already represents some interesting development in this direction. In [33] the actual time evolution equations for the Schmidt orthogonal basis is written down in terms of the "Hamiltonian of the universe", and in all these papers the

emphasis is not so much on "decoherence", but on "stability" of the subsystems. In this way their discussion encompasses more aspects of classicality than just decoherence.

I prefer to mention these questions, but leave them for now. There are more pressing issues, such as the general validity of the Schmidt paths approach which must be addressed before the formalism should be developed further in this direction.

8 Testing the Schmidt paths proposal

There is a major issue which must be resolved if the Schmidt paths proposal is to be taken seriously. In Section 6 I described how the Schmidt paths *ought* to behave in order to describe familiar situations. The question is: "Do the Schmidt paths actually behave as they should?". In order to have even minimal confidence in the proposal one should be able to calculate the Schmidt paths for some familiar, well defined systems, and verify that the paths behave in the correct way. For example, one should be able to construct a "measurement apparatus" which reliably causes the Schmidt orthogonal states of the subsystem to be measured to take on a particular form which is "localized" in the parameter that the apparatus measures. Purely classical subsystems should have Schmidt paths which remain steadily decoherent, and which correctly describe solutions to the classical equations of motion.

I have already started such a testing project, and have seen the Schmidt paths proposal work successfully in some extremely simple situations. The calculations need to be expanded, however, before I would call them a resonable test of the Schmidt paths proposal. I will report these results in another publication once I have expanded the project sufficiently.

One concern I have is that the Schmidt orthogonal decomposition gives a too highly idealized account of the correlations among subsystems. It is perfectly conceivable to me that in making ρ_s totally diagonal the Schmidt orthogonal states must take on some contorted form, whereas some less contorted, but approximate expression of the correlations may come closer to addressing the physically important questions.

On the other hand, the Schmidt orthogonal states *do* appear to provide the clearest statement of the correlations among subsystems, and these cor-

relations appear to be a central feature of any discussion of physics. If the Schmidt paths picture turns out to be wrong, it will be very interesting to find out what aspect of physics takes such a high precedent, as to force the correlations among subsystems to take some secondary (presumably approximate) status.

I hope that others will take up the challenge of testing the Schmidt paths proposal. The computational methods are completely straightforward, and just involve solving the Schrödinger equation and constructing the eigenbases of the various subsystem density matricies. One well thought out example in which the Schmidt orthogonal basis does not behave as it should could be grounds for rejecting this proposal. Whatever the outcome, such an investigation should help clarify the role of correlations in quantum mechanics. The most complicated issue is how one differentiates "over simplified" examples from "realistic" examples. Once more calculations are completed and the mechanisms for decoherence are better understood it should become clearer how problematical this issue really is.

8.1 Continuum problems

A fairly common concern regarding the Schmidt paths relates to continuum systems, such as a pendulum or free particle. One normally thinks of the "classical" states of the pendulum, for example, as being the "coherent states" of the harmonic oscillator. The coherent states do not make an orthonormal basis, so it seems hard to imagine them appearing in the Schmidt orthogonal paths. One thing to remember is that the Schmidt paths for a pendulum need not be too close to coherent states for the proposal to work. They just need to be sufficiently localized to reflect physical reality. In particular, I would expect the extent of localization to depend crucially on the scale on which the environment interacts coherently with the pendulum. For the true "coherent states" the scale of localization just depends on the internal harmonic oscillator parameters, and has nothing to do with any environment.

On a related issue, Joos and Zeh [8] have calculated the Schmidt orthogonal states for a "scattering center" in an environment of photons. These states turn out to be not well localized, and Joos and Zeh conclude that Schmidt orthogonal states are not useful in describing continuum phenomena. However for the calculation they use an "ineffective scattering" approx-

imation, which means they are only considering effects on scales *smaller* than the coherence scale of the interactions. Because they use this approximation I am not convinced of their conclusions.

Unruh[34] has emphasized that the Schmidt orthogonal basis is not well localized in the calculations of reference [11] either. But again, their interaction term also appears to be very coherent over the entire range of positions (q) of the harmonic oscillator they are trying to decohere. The interaction Hamiltonian involves the *same* operator in the environment subspace for all values of q, and only the "effective coupling strength" ϵq varies from one position to another.

I agree that the continuum phenomena mentioned here seem to pose a particular challenge to the Schmidt paths. However, the calculations performed to date are far from conclusive, and the final verdict must await a more thorough analysis.

8.2 Degenerate eigenvalues

It is well known that matricies with degenerate eigenvalues do not have uniquely specified eigenbases. If all the eigenvalues of a matrix are degenerate, the matrix is proportional to the identity matrix, and it is diagonal in *any* basis. Small sets of degenerate eigenvalues cause similar (but less complete) ambiguities. Since the Schmidt orthogonal states are just density matrix eigenstates, eigenvalue degeneracies can cause the Schmidt paths proposal to be poorly defined.

Strictly speaking, in a realistic situation the eigenvalues of ρ_s would only be degenerate at isolated instants in time, as two eigenvalues crossed one another. These situations would involve time varying eigenvalues, which means non-decoherence, and thus the Schmidt orthogonal basis would not have physical importance in these cases. In practice, of course, no eigenvalues will be absolutely constant, but rather they will have some range of small fluctuations. If two neighboring eigenvalues are within the range of small fluctuations of one another, they will cross frequently, and it seems possible that the Schmidt orthogonal basis would evolve in a highly irregular manner.

To some extent the degenerate eigenvalue situation may reflect physical reality. Degenerate eigenvalues in ρ tend to correspond to higher entropy ($\propto -\mathrm{tr}(\rho \log(\rho))$, and it is natural to think that it is harder to identify de-

cohering subsystems in high entropy situations. On the other hand, the formalism should be able to handle without difficulty a "Schrödinger cat" type experiment with two equally probable (but very classical) outcomes. In such a situation the decohering paths should branch with equal probabilities, and that would correspond to some eigenvalues being degenerate.

Kübler and Zeh [33] have calculated the equations of motion for the Schmidt basis vectors. The equations indeed exhibit a striking singularity when eigenvalues of the density matrix are degenerate, indicating rapid time variation. Zeh [25] has made the intriguing conjecture that, in analogy with crossing energy levels, the eigenvalues actually never cross, but repel one another. All the while the different subsystems would remain similarly correlated, but the correlated sets of subsystem states would become associated with different eigenvalues as time evolved. If this situation actually occurred, it would be very interesting, and one might wish to define the Schmidt paths in terms of continuity of the *states* rather than the eigenvalues.

8.3 Degenerate eigenvalues as a larger issue

I have raised the degenerate eigenvalue issue here because I believe it might be a good place to look for trouble in the Schmidt paths proposal. It is interesting to note another place where the degenerate eigenvalue issue has already appeared in the literature on decoherence (although the presence of the corresponding ambiguity was not actually discussed). In one of the classic papers on decoherence Zurek [10], (in equation (2.15)), considers the density matrix of what is *effectively* a two state system. The two states are written in terms of an additional subdivision into a spin (s) and a two state "atom" (a) subspace:

$$\begin{aligned} |1\rangle &= |1\rangle_s \otimes |1\rangle_a \\ |2\rangle &= |2\rangle_s \otimes |2\rangle_a. \end{aligned} \tag{37}$$

(The notation here is slightly different from Zurek's.) Zurek argues that due to correlations with an additional environment system, the density matrix for this two state system will be very nearly diagonal at almost all moments in time.

Zurek focuses particular attention on the case where the diagonal elements of the density matrix are equal. In this case, the density matrix is

(approximately) proportional to the identity, so the correlations with the environment do *not* help one identify a preferred basis, if approximate diagonality of the density matrix is what is sought. The density matrix will be close to diagonal in any basis formed by some unitary transformation of $|1\rangle$ and $|2\rangle$ (defined in Eq (37)).

Still, there is a reason why the states $\{|1\rangle, |2\rangle\}$ form a preferred basis, and that has to do with correlations among the spin and the atom in the *further* subdivision of the $\{|1\rangle, |2\rangle\}$ space. It is only in the $\{|1\rangle, |2\rangle\}$ basis that the the spin and the atom are nicely correlated with one another, as depicted in Eq (37). In any other basis the $|1\rangle_s \otimes |1\rangle_a$ and the $|2\rangle_s \otimes |2\rangle_a$ states would be mixed together.

It seems a bit worrying that in the Schmidt paths case one does not have this sort of flexibility in resolving the ambiguity. That is because unless two eigenvalues are absolutely identical, the Schmidt orthogonal basis is uniquely defined. If two eigenvalues are merely "close" the Schmidt paths scheme allows no choices as to which paths one considers. It remains to be seen if this consideration can cause serious problems in physically relevant situations.

9 Comparison with the decoherence functional approach

Griffiths [20], Omnès[21,22,23], and Gell-Mann and Hartle [24] have considered a "decoherence functional" method for identifying decohering paths within a wavefunction. In this section I compare the Schmidt paths proposal with the decoherence functional approach. I will start by describing the decoherence functional. In keeping with my preferences, I will describe it in the Schrödinger picture, even though it is usually discussed in the Heisenberg picture.

9.1 Review of the decoherence functional

Just as in Schmidt paths scheme, the subdivision of the Hilbert space into subspaces (or coarse graining) is crucial. Unlike in the Schmidt paths case however, the subdivision scheme alone does not specify a particular set of paths to be considered. In the decoherence functional approach *any* path

can be considered. First let us take the simplest case, where there is a single subdivision into "system" and "environment". Specifying a path for the system amounts to specifying a state $|\alpha(t)\rangle_s$ in the system subspace at every moment in time. From this one can construct the projection operators:

$$P_\alpha(t) \equiv |\alpha(t)\rangle\langle\alpha(t)| \otimes \hat{I}_e \tag{38}$$

which project onto the particular path in the subspace and leave the environment unchanged. This path does not specify anything about the environment subsystem, in contrast to the Schmidt paths which also specify the path of the environment.

Part of the procedure is to then choose a particular "coarse graining in time", which amounts to reducing the continuum of projection operators described in Eq (38) to a discrete set. This is done by selecting the projection operators at only a discrete set of times. Having made all these choices, one has a "coarse grained path". It is coarse grained within the Hilbert space due the system/environment subdivision, and coarse grained in time as well. Any change to the choice of path, or either of the coarse graining schemes will result in a different coarse grained path.

Following the notation of Gell-Mann and Hartle I label a particular coarse grained path with $[P_\alpha] = (P^1_{\alpha_1}(t_1), P^2_{\alpha_2}(t_2), ... P^n_{\alpha_n}(t_n))$, representing the corresponding set of projection operators. The α labels the particular path, the integer subscripts label the time slice, and the integer superscripts represent the fact that the P's can be chosen from different sets at different times.

As an intermediate step one constructs what I like to call a "path projected state":

$$|[P_\alpha], \psi_u\rangle \equiv P^n_{\alpha_n} e^{-iH_u(t_n - t_{n-1})} \ldots P^2_{\alpha_2} e^{-iH_u(t_2 - t_1)} P^1_{\alpha_1} e^{-iH_u(t_1 - t_0)} |\psi_u, t_0\rangle \tag{39}$$

This is exactly the state that appears in the discussion surrounding equations (24) and (25) of reference [24]. The decoherence functional, which is a functional of two paths $[P_{\alpha'}]$ and $[P_\alpha]$, is defined as

$$\mathcal{D}([P_{\alpha'}], [P_\alpha]) \equiv \langle [P_{\alpha'}], \psi_u | [P_\alpha] \psi_u \rangle. \tag{40}$$

Typically one wants to consider an "exhaustive set of exclusive alternatives" which means one considers all possible paths made from sets of projection

operators which obey

$$\sum_{\alpha} P_{\alpha}^{k}(t) = 1, \quad P_{\alpha}^{k} P_{\beta}^{k} = \delta_{\alpha\beta} P_{\alpha}^{k} \tag{41}$$

The paths are said to be decohering if the off diagonal elements of $\mathcal{D}$ are close to zero. In that case the diagonal elements of $\mathcal{D}$ give the probabilities associated with the paths.

What one is doing, in this approach, is choosing a particular path, and repeatedly projecting onto it. If the wavefunction describes a subsystem which follows that path in a decohering manner, almost all of the projections will be superfluous. One might visualize the situation by considering a billiard ball rolling along some classical path. One could put down numerous walls its way, but as long as each wall had a doorway on that path, the motion of the ball would be unaffected. However, if you moved just one wall, the ball would not get through. The walls here are meant to represent the projection operators.

If one replaced the billiard ball with an electron, the electron wavefunction would spread out with time, and there would be numerous arrangements of the walls for which there was some probability of the electron getting through. This situation would correspond to many off diagonal elements of $\mathcal{D}$ being non zero, since the path projected states for different paths would have some overlap.

Of course an isolated billiard ball will spread out just as surely (if not as quickly) as an electron. One expects that the decoherence of the billiard ball paths has a lot to do with the different paths being correlated with different environment states, due to local interactions. The Schmidt paths incorporate this feature by choosing the paths based on correlations in the first place. The correlations play a role in the decoherence functional via the $\hat{I}_e$ (the unit operator in the environment subspace) in Eq (38). The matrix element of a P_α between any two states will be zero unless the *environment* parts of the states have some overlap. This allows the path projected states to have little overlap among paths where the subsystem is correlated with orthogonal (or *nearly* orthogonal) environment states. In this way correlations with the environment can cause decoherence as defined in the decoherence functional formalism.

9.2 Possible problems with the decoherence functional

As with the Schmidt paths proposal, I belive the decoherence functional needs to prove itself in some well understood calculable systems. There are a few possible problem areas which I feel should get particular attention. It seems likely that the decoherence functional would accurately represent "truly" decohering paths as such. I am concerned, however, that there are a number of ways it could falsely represent a path as decohering.

The first issue I wish to raise has to do with the coarse graining in time. Any realistic physical subsystem has a some small but finite timescale over which one must wait for the system to exhibit any change (corresponding, loosely, to the largest energy eigenstate for which the system has significant overlap). One can always choose to coarse grain on a much finer timescale than this minimal one, and define paths on which the system is static. Such a set of paths would always be highly decohering according to the decoherence functional, regardless of the true system dynamics. The mathematical content of the situation is identical to the case of the "watchdog effect", where the decay of a system can be prevented by frequent enough measurements (which can be represented by projection operators) [35,36,37].

This extreme example may seem contrived, but I am concerned that this effect may be present to some degree in any decoherence functional calculation. One possible way of addressing this issue would be to vary the coarse graining in time, and require the result to be unaffected.

Another problem could arise when one asks how small is "small enough" for the off diagonal elements of $\mathcal{D}$ to indicate decoherence. Even in cases where there is no decoherence, the off diagonal elements are going to be proportional to a product of complex numbers each with magnitude less than one. The number of factors could well be large, and many of the phases could be un-correlated. Thus, these off diagonal elements will have a tendency to be small in any case. This issue might also be resolved by varying the coarse graining in time, or by some more sophisticated method of determining how small is *really* small. For example in the cases where this "random phase" problem arises, the *on* diagonal elements of $\mathcal{D}$ would also be small, and maybe that can be factored into the discussion. Care would have to be taken, however, since for sufficiently finely grained schemes all the individual diagonal elements of $\mathcal{D}$ will be small.

9.3 Comparison with Schmidt paths

Are the Schmidt paths too rigid?

One striking difference between the decoherence functional approach and the Schmidt paths approach is the number of paths one can consider. In the Schmidt paths approach, once a particular subdivision scheme (or coarse graining) is chosen, the particular paths under consideration are fixed. The paths simply correspond to the terms in the expansion of the wavefunction of the universe in the Schmidt orthogonal basis states, which are uniquely determined. In the decoherence functional approach one is much more flexible, and this flexibility essentially corresponds to the freedom to choose any basis for each of the subsystem. (Note that both approaches have flexibility in choosing which subsystems (or coarse graining scheme) to consider.)

If the Schmidt paths turn out to describe physics correctly, then the lack of flexibility could be regarded as "elegance". This elegance also suggests that the Schmidt paths would make a more powerful tool, in the further qunatification of other aspects of classicality, for example. However, if the Schmidt paths fail the tests discussed in Section 8, then Schmidt paths proposal would appear to be "too rigid", and the flexibility of the decoherence functional would be welcome.

Probability sum rules

As discussed in section 5, the Schmidt paths always have a well defined "instantaneous" probability for which the probability sum rules (relating successive coarse grainings) are exactly obeyed. The issue of decoherence only comes in when one wants to assign a fixed probability over a period of time, and it is this assignment which is approximate. In the decoherence functional approach probabilities are not assigned instantaneously, but only to a whole history. The degree to which the the sum rules are obeyed is approximate, and depends on the degree of decoherence.

Technical problems

In section 9.2 I raised concerns that artifacts could prevent $\mathcal{D}$ from always giving clear and correct answers to questions about decoherence. As I men-

tioned, those issues might well be resolvable, but I do not belive they have been resolved yet. The possible problems that I have mentioned (in Section 8) for the Schmidt paths proposal also raise serious concerns. Both ideas need further investigation, with some concrete calculations, in order to establish their validity (I currently have some of these calculations underway). It is perhaps heartening that, at least at this stage, the technical weak spots for each approach appear to be unrelated, between the two approaches.

Coarse graining in time

The intuitive reason for coarse graining in time is that we should not care about everything that happens on short timescales, as long as it doesn't mess up what is happening on longer, more relevant timescales. In the Schmidt paths proposal the paths are defined at all times, but the idea of temporal coarse graining is incorporated in another way, in the definition of C. The definition (in Section 4) allows for rapid variations in the density matrix eigenvalues over short timescales as long as the amplitude is small, without denying decoherence.

The perspective I take in this paper is that the importance of decoherence, as well as other aspects of classical behavior, is to cause the correlations among subsystems to evolve in a regular way. It is possible that we will learn that C is not the best indicator of this quality (as discussed in Section 7.4). Still, whatever might replace it would probably only demand *long term* regularity in the evolution of correlations. Small, short time fluctuations about a long term trend would no doubt be tolerated. This tolerance would again be reflecting a form of coarse graining in time.

10 Conclusions

I have discussed a well defined proposal for how to identify decohering subsystems within a wavefunction (or density matrix) of an isolated system. This is an important ingredient for identifying classical behavior in fundamentally quantum systems. The starting point is the exact identification of the correlations among subsystems (which is always a precisely defined procedure). In order to attach meaning to these correlations one demands that they evolve in a regular manner. The first step in achieving this regularity is to be able

to assign definite probabilities to different paths, and this feature is know as decoherence. To this end, I define the "coherence" function, $\mathcal{C}$, which must be small for decoherence to occur. Requiring decoherence is shown to be equivalent to demanding unitary evolution of the correlations. Additional aspects of "classicality" are associated with further simplifications of the evolution of correlations.

The "Schmidt paths proposal", which I present in this article, differs in several ways from the "decoherence functional" approach to decoherence. The Schmidt paths proposal is less flexible, and completely specifies the paths to be considered, once a subdivision scheme (or coarse graining) is chosen. This means that it is easier to show that the Schmidt paths proposal is wrong (if that is the case), but if it is correct, the Schmidt paths approach is more elegant, and probably more powerful as well.

I have argued that it is crucial that both the decoherence functional and the Schmidt paths proposal be tested out in explicit calculations of familiar physical systems, and I have pointed to possible weaknesses of both approaches, on which these tests might focus. There have been a few claims that the Schmidt paths scheme is already discredited by certain calculations, but I have discussed why I feel these calculation are not good tests.

The Schmidt paths proposal takes account of the correlations among subsystems in a particularly precise way, and these correlations play a tremendously important role in almost every aspect of physics. If the Schmidt paths approach is shown to be successful, it will expand our understanding of quantum mechanics, particularly as applied to cosmological situations. If the Schmidt paths proposal fails, then by understanding the nature of its failure we will probably still learn an important lesson about the role of correlations in quantum physics.

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